

THE APPROXIMATION PROPERTY FOR SPACES OF LIPSCHITZ FUNCTIONS

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ABSTRACT. Let $\text{Lip}_0(X)$ be the space of all Lipschitz scalar-valued functions on a pointed metric space X . We characterize the approximation property for $\text{Lip}_0(X)$ with the bounded weak* topology using as tools the tensor product, the ϵ -product and the linearization of Lipschitz mappings.

INTRODUCTION

Let (X, d) be a pointed metric space with a base point which we always will denote by 0 and let F be a Banach space. The space $\text{Lip}_0(X, F)$ is the Banach space of all Lipschitz mappings f from X to F that vanish at 0, with the Lipschitz norm defined by

$$\text{Lip}(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

The elements of $\text{Lip}_0(X, F)$ are frequently called Lipschitz operators. If \mathbb{K} is the field of real or complex numbers, $\text{Lip}_0(X, \mathbb{K})$ is denoted by $\text{Lip}_0(X)$. The closed linear subspace of the dual of $\text{Lip}_0(X)$ spanned by the functionals δ_x on $\text{Lip}_0(X)$ with $x \in X$, given by $\delta_x(f) = f(x)$, is a predual of $\text{Lip}_0(X)$. This predual is called the Lipschitz-free space over X and denoted by $\mathcal{F}(X)$ in [10]. We refer the reader to Weaver's book [25] for the basic theory of $\text{Lip}_0(X)$ and its predual $\mathcal{F}(X)$, which is called the Arens–Eells space of X and denoted by $\mathcal{A}(X)$ there.

The study of the approximation property is a topic of interest for many researchers. Let us recall that a Banach space E has the approximation property (in short, (AP)) if for each compact set $K \subset E$ and each $\varepsilon > 0$, there exists a bounded finite-rank linear operator $T: E \rightarrow E$ such that $\sup_{x \in K} \|T(x) - x\| < \varepsilon$. If $\|T\| \leq \lambda$ for some $\lambda \geq 1$, it is said that E has the λ -bounded approximation property (in short, λ -(BAP)).

To our knowledge, little is known about the (AP) for $\text{Lip}_0(X)$. Johnson [17] observed that if X is the closed unit ball of Enflo's space [9], then $\text{Lip}_0(X)$ fails the (AP). Godefroy and Ozawa [11] showed that there exists a compact pointed metric space X such that $\mathcal{F}(X)$ fails the (AP) and hence so does $\text{Lip}_0(X)$. For positive results, $\text{Lip}[0, 1]$ is isomorphic to $L^\infty[0, 1]$ (see [14, p. 224]) and thus $\text{Lip}[0, 1]$ has the (AP). If (X, d) is a doubling compact pointed metric space, in particular a compact subset of a finite dimensional Banach space, and $X^{(\alpha)}$ with $\alpha \in (0, 1)$ denotes the metric space (X, d^α) , then the space $\text{Lip}_0(X^{(\alpha)})$ is isomorphic to ℓ_∞ by [18, Theorem 6.5], and therefore $\text{Lip}_0(X^{(\alpha)})$ has the (AP). In [16] (see also [15]), Johnson proved that $\text{Lip}_0(X)$ has the (AP) if and only if, for each Banach space F , every Lipschitz compact operator from X to F can be approximated in the Lipschitz norm by Lipschitz finite-rank operators.

The most recent research on the (AP) has been directed toward $\mathcal{F}(X)$ rather than on $\text{Lip}_0(X)$. Godefroy and Kalton [10] proved that a Banach space E has the λ -(BAP) if and only if $\mathcal{F}(E)$ has the same property. Lancien and Pernecká [20] showed that $\mathcal{F}(X)$ has the λ -(BAP) whenever X is a doubling metric space. For stronger approximation properties as the existence of finite-dimensional Schauder decompositions or Schauder bases for certain spaces $\mathcal{F}(E)$, one can see the papers of Borel-Mathurin [3],

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Lancien and Pernecká [20] and Hájek and Pernecká [13]. The results in those works provide apparently a limited information about the (AP) for $\text{Lip}_0(X)$ since the (AP) of a Banach space follows from the (AP) of its dual space but the converse does not always hold.

Our aim in this paper is to study the (AP) for the space $\text{Lip}_0(X)$, with the bounded weak* topology. In the seminal paper [1], Aron and Schottenloher initiated the investigation about the (AP) for spaces of holomorphic mappings on Banach spaces. Mujica [21] extended this study to the preduals of such spaces. Their techniques, based on the tensor product, the ϵ -product and the linearization of holomorphic mappings, work just as well for spaces of Lipschitz mappings.

We now describe the contents of this paper. In Section 1, we briefly recall the compact-open topology τ_0 , the approximation property, the ϵ -product and the linearization of Lipschitz mappings.

We address the study of the topology of bounded compact convergence τ_γ on $\text{Lip}_0(X)$ in Section 2. In the terminology of Cooper [5], we prove that τ_γ is the mixed topology $\gamma[\text{Lip}, \tau_0]$ and $(\text{Lip}_0(X), \tau_\gamma)$ is a Saks space. Furthermore, it is shown that τ_γ agrees with the bounded weak* topology τ_{bw^*} .

We give a pair of descriptions of τ_γ by means of seminorms in Section 3. Assuming X is compact, we first identify τ_γ with the classical strict topology β introduced by Buck [4]. A second, and perhaps more interesting, seminorm description for τ_γ motivates the introduction of a new locally convex topology $\gamma\tau_\gamma$ on $\text{Lip}_0(X, F)$.

Section 4 deals with the (AP) for $(\text{Lip}_0(X), \tau_\gamma)$. We identify topologically the space $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$ with the ϵ -product of $(\text{Lip}_0(X), \tau_\gamma)$ and F , and this permits us to prove that the following properties are equivalent:

- (i) $(\text{Lip}_0(X), \tau_\gamma)$ has the (AP).
- (ii) Every Lipschitz operator from X into F can be approximated by Lipschitz finite-rank operators within the topology $\gamma\tau_\gamma$ for all Banach spaces F .
- (iii) $\mathcal{F}(X)$ has the (AP).

In Section 5, we establish a representation of the dual space of $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$.

1. PRELIMINARIES

Topologies on spaces of Lipschitz functions. Let X be a pointed metric space and let E be a Banach space. The compact-open topology or topology of compact convergence on $\text{Lip}_0(X, E)$ is the locally convex topology generated by the seminorms of the form

$$|f|_K = \sup_{x \in K} \|f(x)\|, \quad f \in \text{Lip}_0(X, E),$$

where K varies over the family of all compact subsets of X . We denote by τ_0 the compact-open topology on $\text{Lip}_0(X, E)$, or on any vector subspace of $\text{Lip}_0(X, E)$.

The topology of pointwise convergence on $\text{Lip}_0(X, E)$ is the locally convex topology τ_p generated by the seminorms of the form

$$|f|_F = \sup_{x \in F} \|f(x)\|, \quad f \in \text{Lip}_0(X, E),$$

where F ranges over the family of all finite subsets of X .

Finally, we denote by τ_{Lip} the topology on $\text{Lip}_0(X, E)$ generated by the Lipschitz norm Lip . It is clear that $\tau_p \subset \tau_0$, and the inclusion $\tau_0 \subset \tau_{\text{Lip}}$ follows easily since $|f|_K \leq \text{Lip}(f)\text{diam}(K \cup \{0\})$ for all $f \in \text{Lip}_0(X)$ and each compact set $K \subset X$.

Approximation property and ϵ -product. Let E and F be locally convex Hausdorff spaces. Let $\mathcal{L}(E; F)$ denote the vector space of all continuous linear mappings from E into F , let $\mathcal{L}_b(E; F)$ denote the vector space $\mathcal{L}(E; F)$ with the topology of uniform convergence on the bounded subsets of E and let $\mathcal{L}_c(E; F)$ denote the vector space $\mathcal{L}(E; F)$ with the topology of uniform convergence on the convex balanced compact subsets of E . That last topology coincides with the compact-open topology if the

closed convex hull of each compact subset of E is compact (for example, if E is quasi-complete). When $F = \mathbb{K}$, we write E' instead of $\mathcal{L}(E; \mathbb{K})$, E'_b in place of $\mathcal{L}_b(E; \mathbb{K})$, and E'_c instead of $\mathcal{L}_c(E; \mathbb{K})$. Unless stated otherwise, if E and F are normed spaces, $\mathcal{L}(E; F)$ is endowed with its natural norm topology. Let $E \otimes F$ denote the tensor product of E and F , and $E' \otimes F$ can be identified with the subspace of all finite-rank mappings in $\mathcal{L}(E; F)$.

A locally convex space E is said to have the approximation property (in short, (AP)) if the identity mapping on E lies in the closure of $E' \otimes E$ in $\mathcal{L}_c(E; E)$. This is Schwartz's definition of the (AP) in [23], which is slightly different from Grothendieck's definition in [12], though both definitions coincide for quasi-complete locally convex spaces.

The ϵ -product of E and F , denoted by $E\epsilon F$ and introduced by Schwartz [23, 24], is the space $\mathcal{L}_\epsilon(F'_c; E)$, that is the vector space $\mathcal{L}_\epsilon(F'_c; E)$, with the topology of uniform convergence on the equicontinuous subsets of F' . Notice that if F is a normed space, then equicontinuous sets and norm bounded sets in F' coincide. The topology on $\mathcal{L}_\epsilon(F'_c; E)$ is generated by the seminorms

$$\alpha\epsilon\beta(T) = \sup \{ |\langle T(\mu), \nu \rangle| : \mu \in F', |\mu| \leq \alpha, \nu \in E', |\nu| \leq \beta \}, \quad T \in \mathcal{L}_\epsilon(F'_c; E),$$

where α ranges over the continuous seminorms on F and β over the continuous seminorms on E .

We will use the subsequent results which follow from results of Grothendieck [12], Schwartz [23] and Bierstedt and Meise [2].

Proposition 1.1. [23] *Let E and F be locally convex spaces. Then the transpose mapping $T \mapsto T^t$ from $E\epsilon F$ to $F\epsilon E$ is a topological isomorphism.*

Theorem 1.2. [2, 12, 23] *A locally convex space E has the (AP) if and only if $E \otimes F$ is dense in $E\epsilon F$ for every Banach space F .*

Proposition 1.3. [2, 12, 23] *A locally convex space E has the (AP) if E'_c has the (AP).*

Detailed proofs of the preceding results can be found in the paper [6] by Dineen and Mujica.

Linearization of Lipschitz mappings. The study of the preduals of $\text{Lip}_0(X)$ was approached by Weaver [25] by using a procedure to linearize Lipschitz mappings. A similar process of linearization was presented by Mujica for bounded holomorphic mappings on Banach spaces in [21].

Theorem 1.4. [25] *Let X be a pointed metric space. Then there exist a unique, up to an isometric isomorphism, Banach space $\mathcal{F}(X)$ and an isometric embedding $\delta_X: X \rightarrow \mathcal{F}(X)$ such that*

- (i) $\mathcal{F}(X)$ is the closed linear hull in $\text{Lip}_0(X)'$ of the evaluation functionals $\delta_x: \text{Lip}_0(X) \rightarrow \mathbb{K}$ with $x \in X$, where $\delta_x(g) = g(x)$ for all $g \in \text{Lip}_0(X)$.
- (ii) The Dirac map $\delta_X: X \rightarrow \mathcal{F}(X)$ is the map given by $\delta_X(x) = \delta_x$.
- (iii) For each Banach space E and each $f \in \text{Lip}_0(X, E)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{F}(X); E)$ such that $T_f \circ \delta_X = f$. Furthermore, $\|T_f\| = \text{Lip}(f)$.
- (iv) The evaluation map $f \mapsto T_f$ from $\text{Lip}_0(X, E)$ to $\mathcal{L}(\mathcal{F}(X); E)$, defined by $T_f(\varphi) = \varphi(f)$, is an isometric isomorphism.
- (v) $\text{Lip}_0(X)$ is isometrically isomorphic to $\mathcal{F}(X)'$ via the evaluation map.
- (vi) $\mathcal{F}(X)$ coincides with the space of all linear functionals φ on $\text{Lip}_0(X)$ such that the restriction of φ to the closed unit ball $B_{\text{Lip}_0(X)}$ of $\text{Lip}_0(X)$ is continuous when $B_{\text{Lip}_0(X)}$ is equipped with the topology of pointwise convergence τ_p , and hence with the compact-open topology τ_0 .

The statements (i)–(v) of Theorem 1.4 were proved by Weaver (see [25, Theorem 2.2.4]). The statement (vi) was stated in [15, Lemma 1.1] for τ_p and recall that, by [19, p. 232], the topology τ_p agrees with τ_0 on the equicontinuous subsets of $\text{Lip}_0(X)$, and in particular on $B_{\text{Lip}_0(X)}$.

Viewing $\text{Lip}_0(X)$ as the dual of $\mathcal{F}(X)$, we can consider its weak* topology. We recall that the weak* topology on $\text{Lip}_0(X)$ is the locally convex topology τ_{w^*} generated by the seminorms of the form

$$p_G(f) = \sup_{\varphi \in G} |\varphi(f)|, \quad f \in \text{Lip}_0(X),$$

where G ranges over the family of all finite subsets of $\mathcal{F}(X)$. Let us recall that τ_{w^*} is the smallest topology for $\text{Lip}_0(X)$ such that, for each $\varphi \in \mathcal{F}(X)$, the linear functional $f \mapsto \varphi(f)$ on $\text{Lip}_0(X)$ is continuous with respect to τ_{w^*} .

It is easy to check that $\tau_p \subset \tau_{w^*} \subset \tau_{\text{Lip}}$. Indeed, on a hand, if F is a finite subset of X , then $G = \delta_X(F)$ is a finite subset of $\mathcal{F}(X)$ and

$$|f|_F = \sup_{x \in F} |f(x)| = \sup_{x \in F} |\delta_x(f)| = \sup_{\varphi \in G} |\varphi(f)| = p_G(f)$$

for all $f \in \text{Lip}_0(X)$, and this proves that $\tau_p \subset \tau_{w^*}$. On the other hand, if G is a finite subset of $\mathcal{F}(X)$, then G is a norm bounded subset of $\text{Lip}_0(X)'$ and $p_G(f) \leq \sup_{\varphi \in G} \|\varphi\| \text{Lip}(f)$ for all $f \in \text{Lip}_0(X)$, and this shows that $\tau_{w^*} \subset \tau_{\text{Lip}}$.

The ensuing result was proved by Godefroy and Kalton in [10].

Theorem 1.5. [10] *Let E and F be Banach spaces.*

- (i) *For every mapping $f \in \text{Lip}_0(E, F)$, there exists a unique operator $\widehat{f} \in \mathcal{L}(\mathcal{F}(E); \mathcal{F}(F))$ such that $\widehat{f} \circ \delta_E = \delta_F \circ f$. Furthermore, $\|\widehat{f}\| = \text{Lip}(f)$.*
- (ii) *If E is a subspace of F and $\iota: E \rightarrow F$ is the canonical embedding, then $\widehat{\iota}: \mathcal{F}(E) \rightarrow \mathcal{F}(F)$ is an isometric embedding.*

2. THE TOPOLOGY OF BOUNDED COMPACT CONVERGENCE FOR $\text{Lip}_0(X)$

We recall (see [5, Definition 3.2]) that a Saks space is a triple $(E, \|\cdot\|, \tau)$, where E is a vector space, τ is a locally convex topology on E and $\|\cdot\|$ is a norm on E so that the closed unit ball B_E of $(E, \|\cdot\|)$ is τ -bounded and τ -closed.

Given a pointed metric space X , we consider on $\text{Lip}_0(X)$ the following topologies:

- τ_p : the topology of pointwise convergence.
- τ_0 : the topology of compact convergence.
- τ_{w^*} : the weak* topology $\sigma(\text{Lip}_0(X), \mathcal{F}(X))$.
- τ_{Lip} : the topology of the norm Lip .

The triple $(\text{Lip}_0(X), \text{Lip}, \tau_0)$ is a Saks space since $B_{\text{Lip}_0(X)}$ is τ_0 -compact by the Ascoli theorem (see [19, p. 234]). Then, by [5, 3.4], we can form the mixed topology $\gamma[\text{Lip}, \tau_0]$ on $\text{Lip}_0(X)$. Following [5, Definition 1.4], $\gamma[\text{Lip}, \tau_0]$ is the locally convex topology on $\text{Lip}_0(X)$ generated by the base of neighborhoods of zero $\{\gamma(U)\}$, where $U = \{U_n\}$ is a sequence of convex balanced τ_0 -neighborhoods of zero and

$$\gamma(U) := \bigcup_{n=1}^{\infty} (U_1 \cap B_{\text{Lip}_0(X)} + U_2 \cap 2B_{\text{Lip}_0(X)} + U_3 \cap 2^2 B_{\text{Lip}_0(X)} + \cdots + U_n \cap 2^{n-1} B_{\text{Lip}_0(X)}).$$

Since $\tau_p \subset \tau_{w^*} \subset \tau_0$ on $B_{\text{Lip}_0(X)}$ (the second inclusion follows from Theorem 1.4 (vi)) and $B_{\text{Lip}_0(X)}$ is τ_0 -compact, then $\tau_p = \tau_{w^*} = \tau_0$ on $B_{\text{Lip}_0(X)}$. Then [5, Corollary 1.6] yields

$$\gamma[\text{Lip}, \tau_p] = \gamma[\text{Lip}, \tau_{w^*}] = \gamma[\text{Lip}, \tau_0],$$

and we denote this topology by τ_γ . We gather next some properties of τ_γ .

Theorem 2.1. *Let X be a pointed metric space.*

- (i) τ_0 is smaller than τ_γ , and τ_γ is smaller than τ_{Lip} .
- (ii) τ_γ is the largest locally convex topology on $\text{Lip}_0(X)$ which coincides with τ_0 on each norm bounded subset of $\text{Lip}_0(X)$.

- (iii) If F is a locally convex space and $T: \text{Lip}_0(X) \rightarrow F$ is linear, then T is τ_γ -continuous if and only if $T|_B$ is τ_0 -continuous for each norm bounded subset B of $\text{Lip}_0(X)$.
- (iv) A sequence in $\text{Lip}_0(X)$ is τ_γ -convergent to zero if and only if it is norm bounded and τ_0 -convergent to zero.
- (v) A subset of $\text{Lip}_0(X)$ is τ_γ -bounded if and only if it is norm bounded.
- (vi) A subset of $\text{Lip}_0(X)$ is τ_γ -compact (precompact, relatively compact) if and only if it is norm bounded and τ_0 -compact (precompact, relatively compact).
- (vii) $(\text{Lip}_0(X), \tau_\gamma)$ is a complete semi-Montel space.

Proof. The statements (i)–(vii) follow immediately from the theory of [5, Chapter I]. More namely, (i) and (ii) follow from Proposition 1.5; (iii) from Corollary 1.7; (iv) from Proposition 1.10; (v) from Proposition 1.11; (vi) from Proposition 1.12; and (vii) from Propositions 1.13 and 1.26 and the Ascoli theorem. \square

The property (ii) above justifies the name of topology of bounded compact convergence for τ_γ . We next improve this property.

Theorem 2.2. *Let X be a pointed metric space.*

- (i) τ_γ is the largest topology on $\text{Lip}_0(X)$ which agrees with τ_0 on each norm bounded subset of $\text{Lip}_0(X)$.
- (ii) A subset U of $\text{Lip}_0(X)$ is open (closed) in $(\text{Lip}_0(X), \tau_\gamma)$ if and only if $U \cap B$ is open (closed) in (B, τ_0) for each norm bounded subset B of $\text{Lip}_0(X)$.
- (iii) $\mathcal{F}(X) = (\text{Lip}_0(X), \tau_\gamma)'_b = (\text{Lip}_0(X), \tau_\gamma)'_c$.
- (iv) The evaluation map $f \mapsto T_f$ from $(\text{Lip}_0(X), \tau_\gamma)$ to $\mathcal{F}(X)'_c$ is a topological isomorphism.

Proof. The statements (i) and (ii) follow from [5, Corollary 4.2]. We now prove (iii). From Theorem 1.4 (vi) and Theorem 2.1 (ii)–(iii), we deduce that $\mathcal{F}(X) = (\text{Lip}_0(X), \tau_\gamma)'$ algebraically. Since $\mathcal{F}(X)$ is a linear subspace of $(\text{Lip}_0(X), \tau_{\text{Lip}})'$ by Theorem 1.4 (i) and both spaces $(\text{Lip}_0(X), \tau_{\text{Lip}})$ and $(\text{Lip}_0(X), \tau_\gamma)$ have the same bounded sets by Theorem 2.1 (v), we infer that $\mathcal{F}(X) = (\text{Lip}_0(X), \tau_\gamma)'_b$. The identification $(\text{Lip}_0(X), \tau_\gamma)'_b = (\text{Lip}_0(X), \tau_\gamma)'_c$ follows from the fact that $(\text{Lip}_0(X), \tau_\gamma)$ is a semi-Montel space, and the proof of (iii) is finished.

To prove (iv), notice that $\{nB_{\text{Lip}_0(X)}\}$ is an increasing sequence of convex, balanced and τ_γ -compact subsets of $\text{Lip}_0(X)$ (by Theorem 2.1 (vi) and the Ascoli theorem) with the property that a set $U \subset \text{Lip}_0(X)$ is τ_γ -open whenever $U \cap nB_{\text{Lip}_0(X)}$ is open in $(nB_{\text{Lip}_0(X)}, \tau_\gamma)$ for every $n \in \mathbb{N}$. This property can be proved easily using the statements (i) and (ii). Then, by applying [21, Theorem 4.1], the evaluation map from $(\text{Lip}_0(X), \tau_\gamma)$ to $((\text{Lip}_0(X), \tau_\gamma)'_b)'_c$ is a topological isomorphism. Since $\mathcal{F}(X) = (\text{Lip}_0(X), \tau_\gamma)'_b$ by (iii), the statement (iv) holds. \square

Remark 2.1. *All assertions of Theorems 2.1 and 2.2 are valid if the topology τ_0 is replaced by τ_p or τ_{w^*} .*

We now recall that if E is a Banach space, then the bounded weak* topology on its dual E' , denoted by τ_{bw^*} , is the largest topology on E' agreeing with the topology τ_{w^*} on norm bounded sets [8, V.5.3]. According to the Banach-Dieudonné theorem [8, V.5.4], τ_{bw^*} is just the topology of uniform convergence on sequences in E which tend in norm to zero.

Since $\tau_{w^*} = \tau_0$ on $B_{\text{Lip}_0(X)}$, the assertion (i) of Theorem 2.2 gives the following.

Corollary 2.3. *Let X be a pointed metric space. On the space $\text{Lip}_0(X)$, the bounded weak* topology τ_{bw^*} is the topology τ_γ .*

3. SEMINORM DESCRIPTIONS OF τ_γ ON $\text{Lip}_0(X)$

Our aim in this section is to give a pair of descriptions for τ_γ by means of seminorms. By Theorem 2.2 (ii) and Remark 2.1, the convex balanced sets $U \subset \text{Lip}_0(X)$ such that $U \cap nB_{\text{Lip}_0(X)}$ is a neighborhood of

zero in $(nB_{\text{Lip}_0(X)}, \tau_{w^*})$ for every $n \in \mathbb{N}$, form a base of neighborhoods of zero for τ_γ . For our purposes, we will need the next lemma. For $f \in \text{Lip}_0(X)$ and $A \subset X$, define

$$\text{Lip}_A(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in A, x \neq y \right\}.$$

Notice that if $F \subset X$ is finite, then $\text{Lip}_F(f) = p_G(f)$ (see Section 1) where G is the finite subset of $\mathcal{F}(X)$ given by

$$G = \left\{ \frac{\delta_x - \delta_y}{d(x, y)} : x, y \in F, x \neq y \right\},$$

and hence, for each $\varepsilon > 0$, the set $\{f \in \text{Lip}_0(X) : \text{Lip}_F(f) \leq \varepsilon\}$ is a neighborhood of 0 in $(\text{Lip}_0(X), \tau_{w^*})$.

Lemma 3.1. *Let X be a pointed metric space. Then the sets of the form*

$$U = \bigcap_{n=1}^{\infty} \{f \in \text{Lip}_0(X) : \text{Lip}_{F_n}(f) \leq \lambda_n\},$$

where $\{F_n\}$ is a sequence of finite subsets of X and $\{\lambda_n\}$ is a sequence of positive numbers tending to ∞ , form a base of neighborhoods of zero in $(\text{Lip}_0(X), \tau_\gamma)$.

Proof. We first claim that if $\{F_k\}$ and $\{\lambda_k\}$ are sequences as above, then the set

$$\bigcap_{k=1}^{\infty} \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\}$$

is a neighborhood of 0 in $(\text{Lip}_0(X), \tau_\gamma)$. Indeed, given $n \in \mathbb{N}$, if $m \in \mathbb{N}$ is chosen so that $\lambda_k \geq n$ for $k > m$, then

$$\bigcap_{k=1}^{\infty} \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\} \cap nB_{\text{Lip}_0(X)} = \bigcap_{n=1}^m \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\} \cap nB_{\text{Lip}_0(X)}.$$

The latter is a neighborhood of 0 in $(nB_{\text{Lip}_0(X)}, \tau_{w^*})$, and this proves our claim.

We now must prove that if U is a neighborhood of 0 in $(\text{Lip}_0(X), \tau_\gamma)$, then there are sequences $\{F_k\}$ and $\{\lambda_k\}$ as above for which

$$\bigcap_{k=1}^{\infty} \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\} \subset U.$$

Indeed, we can take a set $U \subset \text{Lip}_0(X)$ such that $U \cap nB_{\text{Lip}_0(X)}$ is an open neighborhood of 0 in $(nB_{\text{Lip}_0(X)}, \tau_{w^*})$ for every $n \in \mathbb{N}$. In particular, $U \cap B_{\text{Lip}_0(X)}$ is a neighborhood of 0 in $(B_{\text{Lip}_0(X)}, \tau_{\text{Lip}})$ and then there exists $\varepsilon > 0$ such that $\varepsilon B_{\text{Lip}_0(X)} \subset U$. In order to prove that there exists a finite set $F_1 \subset X$ such that

$$\{f \in \text{Lip}_0(X) : \text{Lip}_{F_1}(f) \leq \varepsilon\} \cap B_{\text{Lip}_0(X)} \subset U,$$

assume on the contrary that the set

$$\{f \in \text{Lip}_0(X) : \text{Lip}_F(f) \leq \varepsilon\} \cap (B_{\text{Lip}_0(X)} \setminus U)$$

is nonempty for every finite set $F \subset X$. These sets are closed in $(B_{\text{Lip}_0(X)} \setminus U, \tau_{w^*})$ and have the finite intersection property. Since the set $B_{\text{Lip}_0(X)} \setminus U$ is a closed, and therefore compact, subset of $(B_{\text{Lip}_0(X)}, \tau_{w^*})$, we infer that there exists some $f \in B_{\text{Lip}_0(X)} \setminus U$ such that $\text{Lip}_F(f) \leq \varepsilon$ for each finite set $F \subset X$. This implies that $f \in \varepsilon B_{\text{Lip}_0(X)} \setminus U$ which is impossible, and thus proving our assertion.

Proceeding by induction, suppose that we can find finite subsets F_2, \dots, F_n of X such that

$$\bigcap_{k=1}^n \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\} \cap nB_{\text{Lip}_0(X)} \subset U \cap nB_{\text{Lip}_0(X)},$$

where $\lambda_1 = \varepsilon$ and $\lambda_k = k - 1$ for $k > 1$. We will prove that there exists a finite set $F_{n+1} \subset X$ such that

$$\bigcap_{k=1}^{n+1} \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\} \cap (n+1)B_{\text{Lip}_0(X)} \subset U \cap (n+1)B_{\text{Lip}_0(X)}.$$

We argue by contradiction. If no such finite set F_{n+1} exists, then the set

$$C_F := \bigcap_{k=1}^n \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\} \cap \{f \in \text{Lip}_0(X) : \text{Lip}_F(f) \leq n\}$$

has nonempty intersection with the τ_{w^*} -compact set $(n+1)B_{\text{Lip}_0(X)} \setminus U$ for each finite set $F \subset X$. So, by the finite intersection property, there is a $f_0 \in ((n+1)B_{\text{Lip}_0(X)} \setminus U) \cap (\cap_F C_F)$. Therefore $\text{Lip}_F(f_0) \leq n$ for each F and so $\text{Lip}(f_0) \leq n$. Then $f_0 \in U \cap nB_{\text{Lip}_0(X)} \subset U \cap (n+1)B_{\text{Lip}_0(X)}$ which is a contradiction.

Then we can construct, by induction, a sequence $\{F_k\}$ of finite subsets of X so that

$$\bigcap_{k=1}^n \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\} \cap nB_{\text{Lip}_0(X)} \subset U$$

for every $n \in \mathbb{N}$. Since $\text{Lip}_0(X) = \cup_{n=1}^{\infty} nB_{\text{Lip}_0(X)}$, we conclude that

$$\bigcap_{k=1}^{\infty} \{f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k\} \subset U.$$

□

A first characterization of τ_γ by means of seminorms lies over the concept of strict topology, introduced by Buck in [4], for spaces of continuous functions on locally compact spaces.

Let X be a pointed metric space. We denote

$$\tilde{X} = \{(x, y) \in X^2 : x \neq y\}.$$

Let $C_b(\tilde{X})$ be the space of bounded continuous scalar-valued functions on \tilde{X} with the supremum norm, and let Φ be De Leeuw's map from $\text{Lip}_0(X)$ into $C_b(\tilde{X})$ defined by

$$\Phi(f)(x, y) = \frac{f(x) - f(y)}{d(x, y)}.$$

Clearly, Φ is an isometric isomorphism from $\text{Lip}_0(X)$ onto the closed subspace $\Phi(\text{Lip}_0(X))$ of $C_b(\tilde{X})$.

Definition 3.1. Let X be a compact pointed metric space. The strict topology β on $\text{Lip}_0(X)$ is the strict topology on $\Phi(\text{Lip}_0(X))$, that is the locally convex topology generated by the seminorms of the form

$$\|f\|_\phi = \sup_{(x, y) \in \tilde{X}} |\phi(x, y)| \frac{|f(x) - f(y)|}{d(x, y)}, \quad f \in \text{Lip}_0(X),$$

where ϕ runs through the space $C_0(\tilde{X})$ of continuous functions from \tilde{X} into \mathbb{K} which vanish at infinity.

Theorem 3.2. Let X be a compact pointed metric. On the space $\text{Lip}_0(X)$, the strict topology β is the topology τ_γ .

Proof. We first show that the identity is a continuous mapping from $(\text{Lip}_0(X), \tau_\gamma)$ to $(\text{Lip}_0(X), \beta)$. By Theorem 2.1 (iii), it is enough to show that the identity on $nB_{\text{Lip}_0(X)}$ is continuous on $(nB_{\text{Lip}_0(X)}, \tau_0)$ for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and fix $\phi \in C_0(\tilde{X})$ and $\varepsilon > 0$. Then there is a compact set $K \subset \tilde{X}$ such that $|\phi(x, y)| < \varepsilon/2n$ if $(x, y) \in \tilde{X} \setminus K$. Take

$$U = \left\{ f \in \text{Lip}_0(X) : \sup_{(x, y) \in K} \frac{|f(x) - f(y)|}{d(x, y)} \leq \frac{\varepsilon}{2(1 + \|\phi\|_\infty)} \right\}.$$

We now prove that U is a neighborhood of 0 in $(\text{Lip}_0(X), \tau_\gamma)$. Indeed, define $\sigma: \tilde{X} \rightarrow \mathcal{F}(X)$ by

$$\sigma(x, y) = \frac{\delta_x - \delta_y}{d(x, y)}.$$

Since the maps $x \mapsto \delta_x$ and $(x, y) \mapsto d(x, y)$ are continuous, so is also σ . Then $\sigma(K)$ is a compact subset of $\mathcal{F}(X)$ and therefore the polar

$$\sigma(K)^\circ := \left\{ F \in \mathcal{F}(X)': \sup_{(x, y) \in K} |F(\sigma(x, y))| \leq 1 \right\}$$

is a neighborhood of 0 in $\mathcal{F}(X)'_c$. Then, by Theorem 2.2 (iv), the set $\{f \in \text{Lip}_0(X): T_f \in \sigma(K)^\circ\}$, that is

$$\left\{ f \in \text{Lip}_0(X): \sup_{(x, y) \in K} \frac{|f(x) - f(y)|}{d(x, y)} \leq 1 \right\},$$

is a neighborhood of 0 in $(\text{Lip}_0(X), \tau_\gamma)$, and hence so is U as required. It follows that $U \cap nB_{\text{Lip}_0(X)}$ is a neighborhood of 0 in $(nB_{\text{Lip}_0(X)}, \tau_0)$ by Theorem 2.2 (ii). If $f \in U \cap nB_{\text{Lip}_0(X)}$, we have

$$\begin{aligned} \|f\|_\phi &\leq \sup_{(x, y) \in K} |\phi(x, y)| \frac{|f(x) - f(y)|}{d(x, y)} + \sup_{(x, y) \in \tilde{X} \setminus K} |\phi(x, y)| \frac{|f(x) - f(y)|}{d(x, y)} \\ &\leq \|\phi\|_\infty \frac{\varepsilon}{2(1 + \|\phi\|_\infty)} + \frac{\varepsilon}{2n} n < \varepsilon. \end{aligned}$$

Conversely, let U be a neighborhood of 0 in $(\text{Lip}_0(X), \tau_\gamma)$. By Lemma 3.1, we can suppose that

$$U = \bigcap_{n=1}^{\infty} \{f \in \text{Lip}_0(X): \text{Lip}_{F_n}(f) \leq \lambda_n\}$$

where $\{F_n\}$ is a sequence of finite subsets of X and $\{\lambda_n\}$ is a sequence of positive numbers tending to ∞ . We can further suppose that $F_n \subset F_{n+1}$ and $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{N}$. We can construct a function ϕ in $C_0(\tilde{X})$ with $\{(x, y) \in \tilde{X}: \phi(x, y) \neq 0\} \subset \cup_{n=1}^{\infty} F_n$ so that $\phi(x, y) = 1/\lambda_1$ if $(x, y) \in F_1$ and $1/\lambda_{n+1} \leq \phi(x, y) \leq 1/\lambda_n$ for all $(x, y) \in F_{n+1} \setminus F_n$. Then $\{f \in \text{Lip}_0(X): \|f\|_\phi \leq 1\} \subset U$ and this proves the theorem. \square

The second description of τ_γ in terms of seminorms is the ensuing.

Theorem 3.3. *Let X be a pointed metric space. The topology τ_γ is generated by the seminorms of the form*

$$p(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X),$$

where $\{\alpha_n\}$ varies over all sequences in \mathbb{R}^+ tending to zero and $\{(x_n, y_n)\}$ runs over all sequences in \tilde{X} .

Proof. Let \mathcal{V} be the base of neighborhoods of 0 in $(\text{Lip}_0(X), \tau_\gamma)$ formed by the sets of the form

$$U = \bigcap_{n=1}^{\infty} \{f \in \text{Lip}_0(X): \text{Lip}_{F_n}(f) \leq \lambda_n\}$$

where $\{F_n\}$ and $\{\lambda_n\}$ are sequences as in Lemma 3.1. If, for each $U \in \mathcal{V}$, p_U is the Minkowski functional of U , then the family of seminorms $\{p_U: U \in \mathcal{V}\}$ generates the topology τ_γ on $\text{Lip}_0(X)$, but justly we have

$$p_U(f) = \sup_{n \in \mathbb{N}} \lambda_n^{-1} \text{Lip}_{F_n}(f)$$

for all $f \in \text{Lip}_0(X)$, and the result follows. \square

Let E be a Banach space. The polar of $M \subset E$ and the prepolar of $N \subset E'$ are respectively

$$M^\circ = \left\{ f \in E' : \sup_{x \in M} |f(x)| \leq 1 \right\},$$

$$N_\circ = \left\{ x \in E : \sup_{f \in N} |f(x)| \leq 1 \right\}.$$

$\overline{\Gamma}M$ stands for the closed, convex, balanced hull of M in E . The next lemma will be needed later.

Lemma 3.4. *Let X be a pointed metric space. For each compact set $L \subset \mathcal{F}(X)$, there exist sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \tilde{X}^\mathbb{N}$ such that*

$$L \subset \overline{\Gamma} \left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}.$$

Proof. If L is a compact subset of $\mathcal{F}(X)$, then the polar $L^0 := \{F \in \mathcal{F}(X)' : \sup_{\varphi \in L} |F(\varphi)| \leq 1\}$ is a neighborhood of 0 in $\mathcal{F}(X)'_c$. Then, by Theorem 2.2 (iv), the set $\{f \in \text{Lip}_0(X) : T_f \in L^0\}$ is a neighborhood of 0 in $(\text{Lip}_0(X), \tau_\gamma)$. Hence, by Theorem 3.3, there exist sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \tilde{X}^\mathbb{N}$ such that

$$\left\{ f \in \text{Lip}_0(X) : \sup_{n \in \mathbb{N}} \alpha_n \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)} \leq 1 \right\} \subset \left\{ f \in \text{Lip}_0(X) : \sup_{\varphi \in L} |T_f(\varphi)| \leq 1 \right\}.$$

We have

$$\begin{aligned} \left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}^0 &= \left\{ F \in \mathcal{F}(X)' : \sup_{n \in \mathbb{N}} \left| F \left(\alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} \right) \right| \leq 1 \right\} \\ &= \left\{ T_f : f \in \text{Lip}_0(X), \sup_{n \in \mathbb{N}} \alpha_n \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)} \leq 1 \right\} \\ &\subset \left\{ T_f : f \in \text{Lip}_0(X), \sup_{\varphi \in L} |T_f(\varphi)| \leq 1 \right\} \\ &= \left\{ F \in \mathcal{F}(X)' : \sup_{\varphi \in L} |F(\varphi)| \leq 1 \right\} \\ &= L^0, \end{aligned}$$

and then the bipolar theorem yields

$$L \subset (L^0)_0 \subset \left(\left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}^0 \right)_0 = \overline{\Gamma} \left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}.$$

□

4. THE APPROXIMATION PROPERTY FOR $(\text{Lip}_0(X), \tau_\gamma)$

We devote this section to the study of the (AP) for the space $\text{Lip}_0(X)$ with the topology of bounded compact convergence. For it, we introduce the subsequent topology on $\text{Lip}_0(X, F)$.

Definition 4.1. *Let X be a pointed metric and let F be a Banach space. The topology $\gamma\tau_\gamma$ on $\text{Lip}_0(X, F)$ is the locally convex topology generated by the seminorms of the form*

$$q(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X, F),$$

where $\{\alpha_n\}$ ranges over the sequences in \mathbb{R}^+ tending to zero and $\{(x_n, y_n)\}$ over the sequences in \tilde{X} .

We study the relation between the topologies $\gamma\tau_\gamma$, τ_γ and τ_0 .

Proposition 4.1. *Let X be a pointed metric space and let F be a Banach space.*

- (i) τ_0 is smaller than $\gamma\tau_\gamma$ on $\text{Lip}_0(X, F)$.
- (ii) τ_γ agrees with $\gamma\tau_\gamma$ on $\text{Lip}_0(X)$.

Proof. To prove (i), let K be a compact subset of X . Then $\delta_X(K)$ is a compact subset of $\mathcal{F}(X)$ and, by Lemma 3.4, there are sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \widetilde{X}^\mathbb{N}$ such that

$$\delta_X(K) \subset \overline{\Gamma} \left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}.$$

It follows that

$$|f|_K = \sup_{x \in K} \|f(x)\| \leq \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)} = q(f),$$

for all $f \in \text{Lip}_0(X, F)$, as desired. (ii) is deduced from Theorem 3.3 and Definition 4.1. \square

If $f \in \text{Lip}_0(X, F)$, we define the Lipschitz transpose of f to the linear mapping $f^t: F' \rightarrow \text{Lip}_0(X)$ given by $f^t(\psi) = \psi \circ f$ for all $\psi \in F'$. Our next result shows that the Lipschitz transpose can be used to identify the space $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$ with $(\text{Lip}_0(X), \tau_\gamma) \epsilon F$. By Section 1, notice that the seminorms

$$\sup \left\{ \alpha_n \frac{|T(\psi)(x_n) - T(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N}, \psi \in F', \|\psi\| \leq 1 \right\}, \quad T \in (\text{Lip}_0(X), \tau_\gamma) \epsilon F,$$

where $\{\alpha_n\}$ and $\{(x_n, y_n)\}$ are sequences as above, determine the topology of $\mathcal{L}_\epsilon(F'_c; (\text{Lip}_0(X), \tau_\gamma)) = (\text{Lip}_0(X), \tau_\gamma) \epsilon F$.

Theorem 4.2. *Let X be a pointed metric space and let F be a Banach space. The mapping $f \mapsto f^t$ is a topological isomorphism from $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$ onto $(\text{Lip}_0(X), \tau_\gamma) \epsilon F$.*

Proof. If $f \in \text{Lip}_0(X, F)$, the mapping $f^t: F' \rightarrow \text{Lip}_0(X)$ is continuous from F'_c into $(\text{Lip}_0(X), \tau_\gamma)$. To prove this, let p be a continuous seminorm on $(\text{Lip}_0(X), \tau_\gamma)$. By Theorem 3.3, we can suppose that

$$p(g) = \sup_{n \in \mathbb{N}} \alpha_n \frac{|g(x_n) - g(y_n)|}{d(x_n, y_n)}, \quad g \in \text{Lip}_0(X),$$

where $\{\alpha_n\}$ is a sequence in \mathbb{R}^+ tending to zero and $\{(x_n, y_n)\}$ is a sequence in \widetilde{X} . Since

$$\left\| \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\| \leq \alpha_n \text{Lip}(f)$$

for all $n \in \mathbb{N}$, the set

$$K = \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} \cup \{0\}$$

is compact in F . For each $\psi \in F'$, we have

$$\alpha_n \frac{|f^t(\psi)(x_n) - f^t(\psi)(y_n)|}{d(x_n, y_n)} = \left| \psi \left(\alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right) \right| \leq |\psi|_K$$

for all $n \in \mathbb{N}$, and consequently $p(f^t(\psi)) \leq |\psi|_K$ as required.

Clearly, the mapping $f \mapsto f^t$ from $\text{Lip}_0(X, F)$ to $\mathcal{L}_\epsilon(F'_c; (\text{Lip}_0(X), \tau_\gamma))$ is linear and, since F' separates the points of F , is injective. To prove that it is surjective, let $T \in \mathcal{L}_\epsilon(F'_c; (\text{Lip}_0(X), \tau_\gamma))$. Then its transpose T^t is in $\mathcal{L}_\epsilon((\text{Lip}_0(X), \tau_\gamma)'_c; F) = \mathcal{L}_\epsilon(\mathcal{F}(X); F)$ by Proposition 1.1 and Theorem 2.2 (iii). Notice that $T \in \mathcal{L}(F'; \text{Lip}_0(X))$ since the closed unit ball $B_{F'}$ of F' is a compact subset of (F', τ_0) , then $T(B_{F'})$ is a bounded subset of $(\text{Lip}_0(X), \tau_\gamma)$ and hence norm bounded by Theorem 2.1 (v). Consider now the

Dirac map $\delta_X: X \rightarrow \mathcal{F}(X)$. Then the mapping $f = T^t \circ \delta_X$ maps X into F , vanishes at 0 and is Lipschitz since

$$\|f(x) - f(y)\| \leq \|T^t\| \|\delta_X(x) - \delta_X(y)\| = \|T\| d(x, y)$$

for all $x, y \in X$. For every $\psi \in F'$ and $x \in X$, we have

$$f^t(\psi)(x) = \langle \psi, f(x) \rangle = \langle \psi, T^t \delta_X(x) \rangle = \langle T(\psi), \delta_X(x) \rangle = T(\psi)(x),$$

and thus $f^t = T$. Hence the mapping $f \mapsto f^t$ is a linear bijection from $\text{Lip}_0(X, F)$ onto $(\text{Lip}_0(X), \tau_\gamma) \epsilon F$ with inverse given by $T \mapsto T^t \circ \delta_X$.

It remains to show that it is continuous with continuous inverse. For it, let $\{\alpha_n\}$ and $\{(x_n, y_n)\}$ be sequences as above. By Definition 4.1,

$$q(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X, F),$$

is a continuous seminorm on $(\text{Lip}_0(X, F), \gamma \tau_\gamma)$. If $n \in \mathbb{N}$ and $\psi \in F'$ with $\|\psi\| \leq 1$, we have

$$\alpha_n \frac{|f^t(\psi)(x_n) - f^t(\psi)(y_n)|}{d(x_n, y_n)} = \alpha_n \frac{|\psi(f(x_n)) - \psi(f(y_n))|}{d(x_n, y_n)} \leq \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)},$$

therefore

$$\sup \left\{ \alpha_n \frac{|f^t(\psi)(x_n) - f^t(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N}, \psi \in F', \|\psi\| \leq 1 \right\} \leq q(f)$$

and this proves that the mapping $f \mapsto f^t$ is continuous. To see that its inverse $T \mapsto T^t \circ \delta_X$ is continuous, let q be a continuous seminorm on $(\text{Lip}_0(X, F), \gamma \tau_\gamma)$. By definition 4.1, we can suppose that

$$q(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X, F),$$

where $\{\alpha_n\}$ and $\{(x_n, y_n)\}$ are sequences as above. For each $n \in \mathbb{N}$, take $\psi_n \in B_{F'}$ such that

$$\|T^t \delta_X(x_n) - T^t \delta_X(y_n)\| = |\langle \psi_n, T^t \delta_X(x_n) - T^t \delta_X(y_n) \rangle|$$

and then we have

$$\alpha_n \frac{\|T^t \delta_X(x_n) - T^t \delta_X(y_n)\|}{d(x_n, y_n)} = \alpha_n \frac{|\langle T \psi_n, \delta_X(x_n) - \delta_X(y_n) \rangle|}{d(x_n, y_n)} = \alpha_n \frac{|T(\psi_n)(x_n) - T(\psi_n)(y_n)|}{d(x_n, y_n)}.$$

It follows that

$$q(T^t \circ \delta_X) \leq \sup \left\{ \alpha_n \frac{|T(\psi)(x_n) - T(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N}, \psi \in F', \|\psi\| \leq 1 \right\},$$

and the proof is finished. \square

Our next aim is to identify linearly the tensor product $\text{Lip}_0(X) \otimes F$ with the space of all Lipschitz finite-rank operators from X to F . Let us recall that a mapping $f \in \text{Lip}_0(X, F)$ is called a Lipschitz finite-rank operator if the linear hull of $f(X)$ in F has finite dimension in whose case this dimension is called the rank of f and denoted by $\text{rank}(f)$. We represent by $\text{Lip}_{0F}(X, F)$ the vector space of all Lipschitz finite-rank operators from X to F . This space can be generated linearly as follows.

Lemma 4.3. *Let X be a pointed metric space and F a Banach space.*

- (i) *If $g \in \text{Lip}_0(X)$ and $u \in F$, then the map $g \cdot u: X \rightarrow F$, given by $(g \cdot u)(x) = g(x)u$, belongs to $\text{Lip}_{0F}(X, F)$ and $\text{Lip}(g \cdot u) = \text{Lip}(g) \|u\|$. Moreover, $\text{rank}(g \cdot u) = 1$ if $g \neq 0$ and $u \neq 0$.*
- (ii) *Every element $f \in \text{Lip}_{0F}(X, F)$ has a representation in the form $f = \sum_{j=1}^m g_j \cdot u_j$, where $m = \text{rank}(f)$, $g_1, \dots, g_m \in \text{Lip}_0(X)$ and $u_1, \dots, u_m \in F$.*

Proof. (i) Clearly, $g \cdot u$ is well-defined. Let $x, y \in X$. For any $u \in F$, we obtain

$$\|(g \cdot u)(x) - (g \cdot u)(y)\| = \|(g(x) - g(y))u\| = |g(x) - g(y)| \|u\| \leq \text{Lip}(g)d(x, y) \|u\|,$$

and so $g \cdot u \in \text{Lip}_0(X, F)$ and $\text{Lip}(g \cdot u) \leq \text{Lip}(g) \|u\|$. For the converse inequality, note that

$$|g(x) - g(y)| \|u\| = \|(g \cdot u)(x) - (g \cdot u)(y)\| \leq \text{Lip}(g \cdot u)d(x, y)$$

for all $x, y \in X$, and therefore $\text{Lip}(g) \|u\| \leq \text{Lip}(g \cdot u)$.

(ii) Suppose that the linear hull $\text{lin}(f(X))$ of $f(X)$ in F is m -dimensional and let $\{u_1, \dots, u_m\}$ be a base of $\text{lin}(f(X))$. Then, for each $x \in X$, the element $f(x) \in f(X)$ is expressible in a unique form as $f(x) = \sum_{j=1}^m \lambda_j^{(x)} u_j$ with $\lambda_1^{(x)}, \dots, \lambda_m^{(x)} \in \mathbb{K}$. For each $j \in \{1, \dots, m\}$, define the linear map $y^j: \text{lin}(f(X)) \rightarrow \mathbb{K}$ by $y^j(f(x)) = \lambda_j^{(x)}$ for all $x \in X$. Let $g_j = y^j \circ f$. Clearly, $g_j \in \text{Lip}_0(X)$ and $f(x) = \sum_{j=1}^m \lambda_j^{(x)} u_j = \sum_{j=1}^m g_j(x) u_j$ for all $x \in X$. Hence $f = \sum_{j=1}^m g_j \cdot u_j$. \square

Proposition 4.4. *Let X be a pointed metric space and let F be a Banach space. Then $\text{Lip}_0(X) \otimes F$ is linearly isomorphic to $\text{Lip}_{0F}(X, F)$ via the linear bijection $K: \text{Lip}_0(X) \otimes F \rightarrow \text{Lip}_{0F}(X, F)$ given by*

$$K \left(\sum_{j=1}^m g_j \otimes u_j \right) = \sum_{j=1}^m g_j \cdot u_j.$$

Proof. Let $\sum_{j=1}^m g_j \otimes u_j \in \text{Lip}_0(X) \otimes F$. The mapping K is well defined. Indeed, if $\sum_{j=1}^m g_j \otimes u_j = 0$, then $\sum_{j=1}^m \varphi(g_j) u_j = 0$ for every $\varphi \in \text{Lip}_0(X)'$ by [22, Proposition 1.2]. In particular, $\sum_{j=1}^m \delta_x(g_j) u_j = 0$ for every $x \in X$ and thus $\sum_{j=1}^m g_j \cdot u_j = 0$ as required. Clearly, K is linear and, by Lemma 4.3, is onto. To see that it is injective, assume that $K(\sum_{j=1}^m g_j \otimes u_j) = 0$. Then $\sum_{j=1}^m \delta_x(g_j) u_j = 0$ for every $x \in X$, and since $\{\delta_x: x \in X\}$ is a separating subset of $\text{Lip}_0(X)'$, we infer that $\sum_{j=1}^m g_j \otimes u_j = 0$ (see [22, p. 3-4]). \square

From the preceding results we deduce the next result that characterizes the (AP) for the space $\text{Lip}_0(X)$ with the topology of bounded compact convergence.

Corollary 4.5. *Let X be a pointed metric space. The following are equivalent.*

- (i) $(\text{Lip}_0(X), \tau_\gamma)$ has the (AP).
- (ii) $\mathcal{F}(X)$ has the (AP).
- (iii) $\text{Lip}_0(X) \otimes F$ is dense in $(\text{Lip}_0(X), \tau_\gamma) \epsilon F$ for every Banach space F .
- (iv) $\text{Lip}_{0F}(X, F)$ is dense in $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$ for every Banach space F .

Proof. (i) \Leftrightarrow (ii): Assume that $(\text{Lip}_0(X), \tau_\gamma)$ has the (AP). Since $(\text{Lip}_0(X), \tau_\gamma) = F(X)'_c$ by Theorem 2.2 (iv), then $\mathcal{F}(X)$ has the (AP) by Proposition 1.3. Conversely, if $\mathcal{F}(X)$ has the (AP), we use that $\mathcal{F}(X) = (\text{Lip}_0(X), \tau_\gamma)'_c$ by Theorem 2.2 (iii) to obtain that $(\text{Lip}_0(X), \tau_\gamma)$ has the (AP) by Proposition 1.3.

(i) \Leftrightarrow (iii) is an application of Theorem 1.2, and (iii) \Leftrightarrow (iv) follows from Proposition 4.4 and Theorem 4.2. \square

5. THE DUAL SPACE OF $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$

The following theorem describes the dual of the space $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$. Recall that a linear functional T on a topological vector space Y is continuous if and only if there is a neighborhood U of 0 in Y such that $T(U)$ is a bounded subset of \mathbb{K} . Hence $T \in (\text{Lip}_0(X, F), \gamma\tau_\gamma)'$ if and only if there exist a constant $c > 0$ and sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \tilde{X}^\mathbb{N}$ such that

$$|T(f)| \leq c \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}$$

for every $f \in \text{Lip}_0(X, F)$.

Theorem 5.1. *Let X be a pointed metric and let F be a Banach space. Then a linear functional T on $\text{Lip}_0(X, F)$ is in the dual of $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$ if and only if there exist sequences $\{\phi_n\}$ in F' and $\{(x_n, y_n)\}$ in \tilde{X} such that $\sum_{n=1}^{\infty} \|\phi_n\| < \infty$ and*

$$T(f) = \sum_{n=1}^{\infty} \phi_n \left(\frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right)$$

for all $f \in \text{Lip}_0(X, F)$.

Proof. Assume that T is a linear functional on $\text{Lip}_0(X, F)$ of the preceding form. Since $\sum_{n=1}^{\infty} \|\phi_n\| < \infty$, we can take a sequence $\{\lambda_n\}$ in \mathbb{R}^+ tending to ∞ so that $\sum_{n=1}^{\infty} \lambda_n \|\phi_n\| = c < \infty$. Then we have

$$|T(f)| \leq \sum_{n=1}^{\infty} \|\phi_n\| \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)} \leq c \sup_{n \in \mathbb{N}} \lambda_n^{-1} \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}$$

for all $f \in \text{Lip}_0(X, F)$. This proves that T is continuous on $(\text{Lip}_0(X, F), \gamma\tau_\gamma)$.

Conversely, if $T \in (\text{Lip}_0(X, F), \gamma\tau_\gamma)'$, then there are sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \tilde{X}^{\mathbb{N}}$ such that

$$|T(f)| \leq \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}$$

for every $f \in \text{Lip}_0(X, F)$. Consider the linear subspace

$$Z = \left\{ \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} : f \in \text{Lip}_0(X, F) \right\}$$

of $c_0(F)$, and the functional S on Z given by

$$S \left(\left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} \right) = T(f)$$

for every $f \in \text{Lip}_0(X, F)$. It follows easily that S is well defined and linear. Since

$$\left| S \left(\left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} \right) \right| = |T(f)| \leq \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)} = \left\| \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} \right\|_{\infty}$$

for all $f \in \text{Lip}_0(X, F)$, S is continuous on Z . By the Hahn-Banach theorem, S has a norm-preserving continuous linear extension \hat{S} to all of $c_0(F)$. Since $c_0(F)'$ is just $\ell_1(F')$, there exists a sequence $\{\psi_n\}$ in F' such that $\sum_{n=1}^{\infty} \|\psi_n\| = \|\hat{S}\|$ and $\hat{S}(\{u_n\}) = \sum_{n=1}^{\infty} \psi_n(u_n)$ for any $\{u_n\} \in c_0(F)$. Taking $\phi_n = \alpha_n \psi_n$ for each $n \in \mathbb{N}$, we conclude that $\sum_{n=1}^{\infty} \|\phi_n\| \leq \|\{\alpha_n\}\|_{\infty} \|\hat{S}\| < \infty$ and

$$T(f) = \hat{S} \left(\left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} \right) = \sum_{n=1}^{\infty} \phi_n \left(\frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right)$$

for all $f \in \text{Lip}_0(X, F)$. □

Since $\mathcal{F}(X) = (\text{Lip}_0(X), \tau_\gamma)'_b$ by Theorem 2.2 (iii) and $\tau_\gamma = \gamma\tau_\gamma$ on $\text{Lip}_0(X)$ by Proposition 4.1 (ii), we next apply Theorem 5.1 to describe the members of $\mathcal{F}(X)$.

Corollary 5.2. *Let X be a pointed metric. Then $\mathcal{F}(X)$ consists of all functionals $T \in \text{Lip}_0(X)'$ of the form*

$$T(f) = \sum_{n=1}^{\infty} \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X),$$

where $\{\alpha_n\} \in \ell_1$ and $\{(x_n, y_n)\} \in \tilde{X}^{\mathbb{N}}$.

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